

# HYERS-ULAM STABILITY OF THE SPHERICAL FUNCTIONS

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ABSTRACT. In [10] the authors obtained the Hyers-Ulam stability of the functional equation

$$\int_K \int_G f(xtk \cdot y) d\mu(t) dk = f(x)g(y), \quad x, y \in G,$$

where  $G$  is a Hausdorff locally compact topological group,  $K$  is a compact subgroup of morphisms of  $G$ ,  $\mu$  is a  $K$ -invariant complex measure with compact support, provided that the continuous function  $f$  satisfies some Kannappan Type condition. The purpose of this paper is to remove this restriction.

## 1. Introduction

The stability problem of functional equations was posed for the first time by S. M. Ulam [58] in the year 1940. Ulam stated the problem as follows:

Given a group  $G_1$ , a metric group  $(G_2, d)$ , a number  $\varepsilon > 0$  and a mapping  $f: G_1 \rightarrow G_2$  which satisfies the inequality  $d(f(xy), f(x)f(y)) < \varepsilon$  for all  $x, y \in G_1$ , does there exist an homomorphism  $h: G_1 \rightarrow G_2$  and a constant  $k > 0$ , depending only on  $G_1$  and  $G_2$  such that  $d(f(x), h(x)) \leq k\varepsilon$  for all  $x$  in  $G_1$ ?

The first affirmative answer was given by D. H. Hyers [25], under the assumption that  $G_1$  and  $G_2$  are Banach spaces.

In 1978, Th. M. Rassias [43] gave a remarkable generalization of the Hyers's result which allows the Cauchy difference to be unbounded, as follows:

**Theorem 1.1.** [16] *Let  $f: V \rightarrow X$  be a mapping between Banach spaces and let  $p < 1$  be fixed. If  $f$  satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

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for some  $\theta \geq 0$  and for all  $x, y \in V$  ( $x, y \in V \setminus \{0\}$  if  $p < 0$ ). Then there exists a unique additive mapping  $T : V \longrightarrow X$  such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|2 - 2^p|} \|x\|^p$$

for all  $x \in V$  ( $x \in V \setminus \{0\}$  if  $p < 0$ ).

If, in addition,  $f(tx)$  is continuous in  $t$  for each fixed  $x$ , then  $T$  is linear.

Several papers have been published in this subject and some interesting variants of Ulam's problem have been also investigated by a number of mathematicians. We refer the reader to the following references [11], [13], [20], [21], [22], [26]-[46].

The stability of functional equations highlighted a new phenomenon which is now usually called superstability. Consider the functional equation  $E(f) = 0$  and assume we are in a framework where the notion of boundedness of  $f$  and of  $E(f)$  makes sense. We say that the equation  $E(f) = 0$  is superstable if the boundedness of  $E(f)$  implies that either  $f$  is bounded or  $f$  is a solution of  $E(f) = 0$ . This property was first observed when the following theorem was proved by J. Baker, J. Lawrence, and F. Zorzitto [9]

**Theorem 1.2.** *Let  $V$  be a vector space. If a function  $f : V \longrightarrow \mathbb{R}$  satisfies the inequality*

$$|f(x+y) - f(x)f(y)| \leq \varepsilon$$

*for some  $\varepsilon > 0$  and for all  $x, y \in V$ , then either  $f$  is bounded on  $V$  or  $f(x+y) = f(x)f(y)$  for all  $x, y \in V$ .*

The result was generalized by J. A. Baker [8], by replacing  $V$  by a semigroup and  $\mathbb{R}$  by a normed algebra  $E$ , in which the norm is multiplicative, i.e.  $\|uv\| = \|u\|\|v\|$ , for all  $u, v \in E$ , by R. Ger, P. Šemrl [24], where  $E$  is an arbitrary commutative complex semisimple Banach algebra and by J. Lawrence [33] in the case where  $E$  is the algebra of all  $n \times n$  matrices. A different generalization of the result of Baker, Lawrence and Zorzitto was given by L. Székelyhidi [55], [56], [57]. It involves an interesting generalization of the class of bounded function on a group or semigroup. For other superstability results, we can see for example [17], [12], [23], [31], [32] and [48].

Let  $G$  be a Hausdorff locally compact group,  $e$  its identity element. Let  $K$  be a compact subgroup of the group  $\text{Mor}(G)$  of all mappings  $k$  of  $G$  onto itself that are either automorphisms and homeomorphisms ( $k \in K^+$ ), or antiautomorphisms and homeomorphisms ( $k \in K^-$ ). The

action of  $k \in K$  on  $x \in G$  will be denoted by  $k \cdot x$ . Let  $\mu$  be a complex bounded measure on  $G$  with compact support (i.e,  $\mu$  is an element of the topological dual of the Banach spaces of continuous functions vanishing at infinity on  $G$ ).  $\mu$  is assumed to be a  $K$ -invariant measure that is,  $\int_G f(k \cdot t) d\mu(t) = \int_G f(t) d\mu(t)$ , for all  $k \in K$  and for all continuous complex valued function  $f$  on  $G$ .

The main purpose of this paper is to investigate the Hyers-Ulam stability of the functional equations

$$(1.1) \quad \int_G \int_K f(xtk \cdot y) d\mu(t) dk = f(x)g(y), \quad x, y \in G.$$

Indeed we prove the superstability theorem of the functional equation

$$(1.2) \quad \int_G \int_K f(xtk \cdot y) d\mu(t) dk = f(x)f(y), \quad x, y \in G.$$

The functional equation (1.1) is a generalization of many functional equations. The functional equation (1.2) with  $\mu = \delta_e$ : Dirac measure concentrated on the identity element of  $G$  reduce to  $K$ -spherical functions:

$$(1.3) \quad \int_K f(xk \cdot y) dk = f(x)f(y), \quad x, y \in G.$$

The  $K$ -spherical functions and related equations has been widely studied by H. Stetkær see for example [53] and [54]. The bounded solutions of  $K$ -spherical functions in an abelian group are obtained by W. Chojnacki [16] and later by Badora [5] while Stetkær [50], [49] studied unbounded solutions. In [47] H. Shin'ya described all continuous solutions of (1.3) for abelian group. The functional equation (1.2) is considered in [18], [19] and [5]. The functional equation

$$(1.4) \quad \int_K f(xk \cdot y) dk = f(x)g(y), \quad x, y \in G$$

has been examined in special cases by many mathematicians. These cases for example include the cosine equation or d'Alembert's functional equations (cf. [1], [51], [52]...)

$$(1.5) \quad f(x+y) + f(x-y) = 2f(x)f(y), \quad x, y \in G,$$

Wilson's functional equation

$$(1.6) \quad f(x+y) + f(x-y) = 2f(x)g(y), \quad x, y \in G,$$

where  $K = \{Id, -Id\}$  and the Cauchy's equation

$$(1.7) \quad f(x+y) = f(x)f(y), \quad x, y \in G,$$

with  $K = \{ Id \}$ .

During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability of a special case of the functional equation (1.3) and its generalization (1.4). In [6] R. Badora obtained the Hyers-Ulam stability of equation (1.4), where  $G$  is abelian and  $K \subseteq Aut(G)$ : the group of automorphisms of  $G$ .

We note here that the results of R. Badora [6] are also corrects in the case where  $G$  is not necessarily abelian and  $K \subseteq Aut(G)$ . Other results of stability of functional equations related to  $K$ -spherical functions where studied in [2], [3], [4], [14] and [15].

In [10] B. Bouikhalene and E. Elqorachi obtained the Hyers-Ulam stability of the functional equation (1.1), provided that the continuous function  $f$  satisfies the Kannappan type condition:

$$\int_G \int_G f(ztxsy) d\mu(t) d\mu(t) = \int_G \int_G f(ztyxs) d\mu(t) d\mu(t)$$

for all  $x, y, z \in G$ . The purpose of this paper is to remove this restriction.

Throughout this paper,  $G$  is a locally compact group (not necessarily abelian)  $K$  is a compact subgroup of morphisms of  $G$  and  $\mu$  is a complex measure with compact support and which is  $K$ -invariant.

## 2. HYERS ULAM STABILITY OF EQUATION (1.1)

In this section, we will investigate the Hyers Ulam stability of equation (1.1). The following lemma will be helpful in the sequel.

**Lemma 2.1.** *Let  $f: G \rightarrow \mathbb{C}$  be a continuous function. Let  $\mu$  be a complex measure with compact support and which is  $K$ -invariant. Then*

$$\begin{aligned} (2.1) \quad & \int_G \int_K \int_K \int_G f(zth \cdot (k \cdot ysx)) d\mu(t) dhdk d\mu(s) + \int_G \int_K \int_K \int_G f(zth \cdot (xsk \cdot y)) d\mu(t) dhdk d\mu(s) dk \\ &= \int_G \int_K \int_K \int_G f(ztk \cdot ysh \cdot x) d\mu(t) dhdk d\mu(s) + \int_G \int_K \int_K \int_G f(zth \cdot xsk \cdot y) d\mu(t) dhdk d\mu(s) \end{aligned}$$

for all  $x, y, z \in G$ .

*Proof.*

$$\int_G \int_K \int_K \int_G f(zth \cdot (k \cdot ysx)) d\mu(t) dhdk d\mu(s) + \int_G \int_K \int_K \int_G f(zth \cdot (xsk \cdot y)) d\mu(t) dhdk d\mu(s) dk$$

$$\begin{aligned}
(2.2) \quad &= \int_G \int_{K^+} \int_K \int_G f(zthk \cdot yh \cdot sh \cdot x) d\mu(t) dh dk d\mu(s) + \int_G \int_{K^-} \int_K \int_G f(zth \cdot xh \cdot shk \cdot y) d\mu(t) dh d\mu(s) dk \\
&+ \int_G \int_{K^+} \int_K \int_G f(zth \cdot xh \cdot shk \cdot y) d\mu(t) dh dk d\mu(s) + \int_G \int_{K^-} \int_K \int_G f(zthk \cdot yh \cdot sh \cdot x) d\mu(t) dh d\mu(s) dk
\end{aligned}$$

Since  $\mu$  is  $K$ -invariant and the Haar measure  $dk$  is invariant, then we get

$$\begin{aligned}
(2.3) \quad &\int_G \int_{K^+} \int_K \int_G f(zthk \cdot yh \cdot sh \cdot x) d\mu(t) dh dk d\mu(s) = \int_G \int_{K^+} \int_K \int_G f(ztk \cdot ysh \cdot x) d\mu(t) dh dk d\mu(s) \\
&\int_G \int_{K^-} \int_K \int_G f(zth \cdot xh \cdot shk \cdot y) d\mu(t) dh d\mu(s) dk = \int_G \int_{K^-} \int_K \int_G f(zth \cdot xsk \cdot y) d\mu(t) dh d\mu(s) dk \\
&\int_G \int_{K^+} \int_K \int_G f(zth \cdot xh \cdot shk \cdot y) d\mu(t) dh dk d\mu(s) = \int_G \int_{K^+} \int_K \int_G f(zth \cdot xsk \cdot y) d\mu(t) dh dk d\mu(s) \\
&\int_G \int_{K^-} \int_K \int_G f(zthk \cdot yh \cdot sh \cdot x) d\mu(t) dh d\mu(s) dk = \int_G \int_{K^-} \int_K \int_G f(ztk \cdot ysh \cdot x) d\mu(t) dh d\mu(s) dk.
\end{aligned}$$

Now, by using (2.2) and (2.3), we obtain

$$\begin{aligned}
&\int_G \int_K \int_K \int_G f(zth \cdot (k \cdot ysx)) d\mu(t) dh dk d\mu(s) + \int_G \int_K \int_K \int_G f(zth \cdot (xsk \cdot y)) d\mu(t) dh d\mu(s) dk \\
&= \int_G \int_{K^+} \int_K \int_G f(ztk \cdot ysh \cdot x) d\mu(t) dh dk d\mu(s) + \int_G \int_{K^-} \int_K \int_G f(ztk \cdot ysh \cdot x) d\mu(t) dh d\mu(s) dk \\
&+ \int_G \int_{K^-} \int_K \int_G f(zth \cdot xsk \cdot y) d\mu(t) dh d\mu(s) dk + \int_G \int_{K^+} \int_K \int_G f(zth \cdot xsk \cdot y) d\mu(t) dh dk d\mu(s) \\
&= \int_G \int_K \int_K \int_G f(ztk \cdot ysh \cdot x) d\mu(t) dh dk d\mu(s) + \int_G \int_K \int_K \int_G f(zth \cdot xsk \cdot y) d\mu(t) dh dk d\mu(s),
\end{aligned}$$

which gives equation (2.1).  $\square$

The main result of this section

**Theorem 2.2.** *Let  $\delta > 0$ . Suppose that the continuous functions  $f, g: G \rightarrow \mathbb{C}$  satisfy the inequality*

$$(2.4) \quad \left| \int_K \int_G f(xtk \cdot y) d\mu(t) dk - f(x)g(y) \right| < \delta$$

for all  $x, y \in G$ . Then,

i)  $f, g$  are bounded or

ii)  $f$  is unbounded and  $g$  satisfies the functional equation

$$(2.5) \quad \int_K \int_G g(xtk \cdot y) d\mu(t) dk + \int_K \int_G g(k \cdot ytx) d\mu(t) dk = 2g(x)g(y), \quad x, y \in G$$

or iii)  $g$  is unbounded,  $f$  satisfies (1.1) (if  $f \neq 0$ , then  $g$  satisfies equation (2.5)).

*Proof.* Assume that  $f, g$  are continuous and satisfy inequality (2.4). In the first case, we suppose that  $f$  is unbounded. Then from (2.4) we get

$$\begin{aligned}
& \left| \int_G \int_K \int_K \int_G f(zth \cdot (xsk \cdot y)) d\mu(t) dh dk d\mu(s) - f(z) \int_G \int_K g(xsk \cdot y) dk d\mu(s) \right| \\
& \leq \int_G \int_K \left| \int_K \int_G f(zth \cdot (xsk \cdot y)) d\mu(t) dh - f(z) g(xsk \cdot y) |d\mu|(s) dk \right| \leq \delta \|\mu\| \\
& \left| \int_G \int_K \int_K \int_G f(zth \cdot (k \cdot ysx)) d\mu(t) dh dk d\mu(s) - f(z) \int_G \int_K g(k \cdot ysx) dk d\mu(s) \right| \leq \delta \|\mu\| \\
& \left| \int_G \int_K \int_K \int_G f(zth \cdot xsk \cdot y) d\mu(t) dh dk d\mu(s) - \int_G \int_K f(zth \cdot x) dh d\mu(t) g(y) \right| \leq \delta \|\mu\| \\
& \left| \int_G \int_K \int_K \int_G f(zth \cdot ysk \cdot x) d\mu(t) dh dk d\mu(s) - \int_G \int_K f(zth \cdot y) dh d\mu(t) g(x) \right| \leq \delta \|\mu\|.
\end{aligned}$$

So, by using lemma 2.1, the triangle inequality, we obtain

$$\begin{aligned}
& |f(z)| \left| 2g(x)g(y) - \int_G \int_K g(xtk \cdot y) d\mu(t) dk - \int_G \int_K g(k \cdot ytx) d\mu(t) dk \right| \\
& \leq \left| \int_G \int_K \int_G \int_K f(zsh \cdot (xtk \cdot y)) d\mu(t) d\mu(s) dh dk - f(z) \int_G \int_K g(xtk \cdot y) d\mu(t) dk \right| \\
& + \left| \int_G \int_K \int_G \int_K f(zsh \cdot (k \cdot ytx)) d\mu(t) d\mu(s) dh dk - f(z) \int_G \int_K g(k \cdot ytx) d\mu(t) dk \right| \\
& + \left| \int_G \int_K \int_G \int_K f(zsh \cdot xtk \cdot y) d\mu(t) d\mu(s) dh dk - \int_G \int_K f(zsh \cdot x) d\mu(s) dh g(y) \right| \\
& + \left| \int_G \int_K \int_G \int_K f(zsh \cdot ytk \cdot x) d\mu(t) d\mu(s) dh dk - \int_G \int_K f(zsk \cdot y) d\mu(s) dk g(x) \right| \\
& + |g(y)| \left| \int_G \int_K f(zth \cdot x) d\mu(t) dh - f(z) g(x) \right| + |g(x)| \left| \int_G \int_K f(zth \cdot y) d\mu(t) dh - f(z) g(y) \right|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |f(z)| \left| 2g(x)g(y) - \int_G \int_K g(xtk \cdot y) d\mu(t) dk - \int_G \int_K g(k \cdot ytx) d\mu(t) dk \right| \\
& \leq \delta \|\mu\| + \delta \|\mu\| + \delta \|\mu\| + \delta \|\mu\| + |g(y)|\delta + |g(x)|\delta.
\end{aligned}$$

Since  $f$  is assumed to be unbounded, then we get

$$2g(x)g(y) - \int_G \int_K g(xtk \cdot y) d\mu(t) dk - \int_G \int_K g(k \cdot ytx) d\mu(t) dk = 0$$

for all  $x, y \in G$ . This proves the case ii). Now, assume that  $g$  is unbounded. It's easily verified that  $f = 0$  satisfies equation (1.1). For latter, we suppose that  $f \neq 0$ . From inequality (2.4) and the triangle

inequality, we conclude that  $f$  is also unbounded, then from the case ii) the function  $g$  satisfies equation (2.5). For all  $x, y, z \in G$ , we have

$$\begin{aligned}
& |g(z)| \left| \int_G \int_K f(xtk \cdot y) d\mu(t) dk - f(x)g(y) \right| \\
& \leq \left| \int_G \int_K \int_G \int_K f(xtk \cdot ysh \cdot z) d\mu(t) d\mu(s) dh dk - \int_G \int_K f(xtk \cdot y) d\mu(t) dk g(z) \right| \\
& + \left| \int_G \int_K \int_G \int_K f(xth \cdot (ysk \cdot z)) d\mu(t) d\mu(s) dh dk - f(x) \int_G \int_K g(ysk \cdot z) d\mu(s) dk \right| \\
& + \left| \int_G \int_K \int_G \int_K f(xth \cdot (k \cdot zsy)) d\mu(t) d\mu(s) dh dk - f(x) \int_G \int_K g(k \cdot zsy) d\mu(s) dk \right| \\
& + \left| \int_G \int_K \int_G \int_K f(xtk \cdot zsh \cdot y) d\mu(t) d\mu(s) dh dk - \int_G \int_K f(xtk \cdot z) d\mu(t) dk g(y) \right| \\
& + |g(y)| \left| \int_G \int_K f(xtk \cdot z) d\mu(t) dk - f(x)g(z) \right| \\
& + |f(x)| \left| \int_G \int_K g(ysk \cdot z) d\mu(s) dk + \int_G \int_K g(k \cdot zsy) d\mu(s) dk - 2g(y)g(z) \right| \\
& \leq 4\delta \|\mu\| + |g(y)|\delta + |f(x)| \times 0 = 4\delta \|\mu\| + |g(y)|\delta.
\end{aligned}$$

Since  $g$  is unbounded, then  $f$  satisfies equation (1.1). This completes the proof.  $\square$

By using the above result, we get the following corollary.

**Corollary 2.3.** (Superstability of equation (1.2)) Let  $\delta > 0$ . If a continuous function  $f: G \rightarrow \mathbb{C}$  satisfies the inequality

$$(2.6) \quad \left| \int_G \int_K f(xtk \cdot y) d\mu(t) dk - f(x)f(y) \right| \leq \delta, \quad x, y \in G.$$

Then either

$$(2.7) \quad |f(x)| \leq \frac{\|\mu\| + \sqrt{\|\mu\|^2 + 4\delta}}{2}, \quad x \in G$$

or

$$(2.8) \quad \int_G \int_K f(xtk \cdot y) d\mu(t) dk = f(x)f(y), \quad x, y \in G.$$

**Corollary 2.4.** (Superstability of the classical d'Alembert's functional equation) Let  $\mu = \delta_e$ . Let  $\delta > 0$ . Let  $\sigma: G \rightarrow G$  be an involution of  $G$  ( $\sigma(xy) = \sigma(y)\sigma(x)$  and  $\sigma(\sigma(x)) = x$  for all  $x, y \in G$ ). Let  $K = \{I, \sigma\}$ . If a function  $f: G \rightarrow \mathbb{C}$  satisfies the inequality

$$(2.9) \quad |f(xy) + f(x\sigma(y)) - 2f(x)f(y)| \leq \delta, \quad x, y \in G.$$

Then either

$$(2.10) \quad |f(x)| \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, x \in G$$

or

$$(2.11) \quad f(xy) + f(x\sigma(y)) = 2f(x)f(y), x, y \in G.$$

The following general corollary holds on any group and for  $K \subseteq \text{Mor}(G)$ . It's a generalization of the result obtained by Badora in [6].

**Corollary 2.5.** *Let  $\mu = \delta_e$ . Let  $\delta > 0$ . Suppose that the continuous function  $f: G \rightarrow \mathbb{C}$  satisfy the inequality*

$$(2.12) \quad \left| \int_K f(xk \cdot y) dk - f(x)g(y) \right| < \delta, x, y \in G.$$

Then,

- i)  $f, g$  are bounded or
- ii)  $f$  is unbounded and  $g$  satisfies the functional equation

$$(2.13) \quad \int_K g(xk \cdot y) dk + \int_K g(k \cdot yx) dk = 2g(x)g(y), x, y \in G$$

or iii)  $g$  is unbounded,  $f$  satisfies (1.4) (if  $f \neq 0$ , then  $g$  satisfies equation (2.13)).

**Corollary 2.6.** [6] *Let  $\mu = \delta_e$ ,  $K \subseteq \text{Aut}(G)$ . Let  $\delta > 0$ . Suppose that the continuous functions  $f, g: G \rightarrow \mathbb{C}$  satisfy the inequality*

$$(2.14) \quad \left| \int_K f(xk \cdot y) dk - f(x)g(y) \right| < \delta, x, y \in G.$$

Then,

- i)  $f, g$  are bounded or
- ii)  $f$  is unbounded and  $g$  satisfies the functional equation

$$(2.15) \quad \int_K g(xk \cdot y) dk = g(x)g(y), x, y \in G$$

or iii)  $g$  is unbounded,  $f$  satisfies (1.4) (if  $f \neq 0$ , then  $g$  satisfies equation (2.15)).

**Corollary 2.7.** *Let  $\delta > 0$ . Let  $K = \{Id, \sigma\}$ , where  $\sigma$  is an involution of  $G$ .  $\mu$  is a complex measure with compact support and which is  $\sigma$ -invariant. Suppose that the continuous functions  $f, g: G \rightarrow \mathbb{C}$  satisfy the inequality*

$$(2.16) \quad \left| \int_G f(xty) d\mu(t) + \int_G f(xt\sigma(y)) d\mu(t) - 2f(x)g(y) \right| < \delta, x, y \in G.$$



Then,

i)  $f, g$  are bounded or

ii)  $f$  is unbounded and  $g$  satisfies the functional equation

(2.17)

$$\int_G g(xty)d\mu(t) + \int_G g(ytx)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 4g(x)g(y)$$

or iii)  $g$  is unbounded,  $f$  satisfies

$$(2.18) \quad \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y), x, y \in G$$

(if  $f \neq 0$ , then  $g$  satisfies equation (2.17)).

The following corollary is a generalization of the result obtained by E. Elqorachi and M. Akkouchi in [17] under the condition that  $f$  satisfies the Kannappan type condition or  $\mu$  is a generalized Gelfand measure.

**Corollary 2.8.** *Let  $\delta > 0$ . Let  $K = \{Id, \sigma\}$ , where  $\sigma$  is an involution of  $G$ .  $\mu$  is a complex measure with compact support and which is  $\sigma$ -invariant. Suppose that the continuous functions  $f, g : G \rightarrow \mathbb{C}$  satisfy the inequality*

(2.19)

$$\left| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)f(y) \right| < \delta, x, y \in G.$$

Then either

$$(2.20) \quad |f(x)| \leq \frac{\|\mu\| + \sqrt{\|\mu\|^2 + 2\delta}}{2}, x \in G$$

or

$$(2.21) \quad \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)f(y), x, y \in G$$

## REFERENCES

- [1] J. Aczél and J. Dhombres, Functional equations in several variables. With applications to mathematics, information theory and to the natural and social sciences. Encyclopedia of Mathematics and its Applications, 31. Cambridge University Press, Cambridge, 1989.
- [2] R. Badora, Stability Properties of Some Functional Equations. In: Themistocles Rassias, Janusz Brzdek (ed.) Functional Equations in Mathematical Analysis, pp.3-13. Springer Optimization and Its Applications 52, 2011.
- [3] R. Badora, On the stability of a functional equation for generalized trigonometric functions. In: Th.M. Rassias (ed.) Functional Equations and Inequalities, pp.1-5. Kluwer Academic Publishers, 2000.

- [4] R. Badora, On the stability of some functional equations. In: Report of Meeting, 10th International Conference on Functional Equations and Inequalities (September 11-17, 2005, Bedlewo, Poland), p.130. Ann. Acad. Paed. Cracoviensis Studia Math. 5(2006)
- [5] R. Badora, On a joint generalization of Cauchy's and d'Alembert's functional equations. Aequationes Math. 43(1992), No. 1, 7289.
- [6] Badora R., On Hyers-Ulam stability of Wilson's functional equation, Aequationes Math. 60(2000), 211-218.
- [7] R. Badora, Note on the superstability of the Cauchy functional equation, Publicationes Mathematicae Debrecen, vol. 57(2000), No. 3-4, 421-424.
- [8] Baker J. A., The stability of the cosine equation, Proc. Amer. Math. Soc. 80(1980), 411-416.
- [9] Baker J., Lawrence J. and Zorzitto F., The stability of the equation  $f(x + y) = f(x)f(y)$ , Proc. Amer. Math. Soc. 74(1979), 242-246.
- [10] B. Bouikhalene and E. Elqorachi, On Stetkær type functional equations and Hyers-Ulam stability, Publicationes Math. 69(2006), No.1-2(6).
- [11] B. Bouikhalene, E. Elqorachi and Th. M. Rassias, On the Hyers-Ulam stability of approximately Pexider mappings. Math. Inequal. Appl., 11(2008), 805-818.
- [12] B. Bouikhalene, E. Elqorachi, and J. M. Rassias, The superstability of d'Alembert's functional equation on the Heisenberg group, Applied Mathematics Letters, 23(2000), No.1, 105-109.
- [13] L. Cădariu and V. Radu, Fixed points and the stability of Jensen's functional equation, Journal of Inequalities in Pure and Applied Mathematics, 4(2003), No. 3 article 4.
- [14] A. Charifi, B. Bouikhalene and E. Elqorachi, Hyers-Ulam-Rassias stability of a generalized Pexider functional equation, Banach J. Math. Anal., 1(2007), 176-185.
- [15] A. Charifi, B. Bouikhalene, E. Elqorachi and A. Redouani, Hyers-Ulam-Rassias Stability of a generalized Jensen functional equation, Australian J. Math. Anal. Appl. 19(2009), 1-16.
- [16] W. Chojnacki, On some functional equation generalizing Cauchy's and d'Alembert's functional equations. Colloq. Math. 55(1988), No. 1, 169-178.
- [17] Elqorachi E. and Akkouchi M., The superstability of the generalized d'Alembert functional equation, Georgian Math. J. 10(2003), 503-508.
- [18] E. Elqorachi, M. Akkouchi, A. Bakali, and B. Bouikhalene, Badora's equation on non-abelian locally compact groups. Georgian Math. J. 11(2004), No. 3, 449-466.
- [19] E. Elqorachi, M. Akkouchi and B. Bouikhalene. Functional Equation and  $\mu$ -Spherical Functions. Georgian Math. J. 15 (2008), No. 1, 1-20
- [20] Forti, G. L., Hyers-Ulam stability of functional equations in several variables. Aequationes Math. 50 (1995), 143-190.
- [21] Gajda, Z., On stability of additive mappings, *Internat. J. Math. Sci.* **14** (1991), 431-434.
- [22] Găvruta, P. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. 184 (1994), 431-436.
- [23] Ger R., Superstability is not natural, Rocznik Nauk.-Dydakt. Prace Mat. 159 (1993), No. 13, 109-123.

- [24] Ger R. and Šemrl P., The stability of the exponential equation, *Proc. Amer. Soc.* V. 124 (1996), 779-787.
- [25] Hyers, D. H., On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U. S. A.* 27(1941), 222-224.
- [26] Hyers, D. H. and Rassias, Th. M., *Approximate homomorphisms*, *Aequationes Math.*, 44(1992), 125-153.
- [27] Hyers, D. H., Isac, G. I. and Rassias, Th. M., *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
- [28] Jung, S.-M., Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, *Hadronic Press, Inc., Palm Harbor, Florida*, 2003.
- [29] Jung, S.-M., Stability of the quadratic equation of Pexider type, *Abh. Math. Sem. Univ. Hamburg*, 70(2000), 175-190.
- [30] Jung, S.-M., Sahoo, P. K., Stability of a functional equation of Drygas, *Aequationes Math.* 64(2002), No. 3, 263-273.
- [31] G. H. Kim, On the stability of trigonometric functional equations, *Advances in Difference Equations*, vol. 2007, Article ID 90405, 10 pages.
- [32] G. H. Kim, On the stability of the Pexiderized trigonometric functional equation, *Applied Mathematics and Computation*, 203(2008), No. 1, 99-105.
- [33] J. Lawrence, The stability of multiplicative semigroup homomorphisms to real normed algebras, *Aequationes Math.* 28 (1985), 94-101.
- [34] M. S. Moslehian, The Jensen functional equation in non-Archimedean normed spaces, *J. Funct. Spaces Appl.*, 7 (2009), 13-24.
- [35] M. S. Moslehian and Gh. Sadeghi, Stability of linear mappings in quasi-Banach modules, *Math. Inequal. Appl.*, 11 (2008), 549-557.
- [36] A. Najati and M. B. Moghimi, Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces, *J. Math. Anal. Appl.*, 337 (2008), 399-415.
- [37] A. Najati, On the stability of a quartic functional equation, *J. Math. Anal. Appl.*, 340 (2008), 569-574.
- [38] A. Najati and C. Park, Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation, *J. Math. Anal. Appl.*, 335 (2007), 763-778.
- [39] M. M. Pourpasha, J. M. Rassias, R. Saadati and S. M. Vaezpour, A fixed point approach to the stability of Pexider quadratic functional equation with involution, *J. Ineq. Appl.* (2010) Article ID 839639, doi:10.1155/2010/839639.
- [40] J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.*, 46(1982), 126-130.
- [41] J. M. Rassias, Solution of a problem of Ulam, *J. Approx. Theory.*, 57(1989), 268-273.
- [42] J. M. Rassias, On the Ulam stability of mixed type mappings on restricted domains, *J. Math. Anal. Appl.*, 276 (2002), No.2, 747-762.
- [43] Rassias, Th. M., *On the stability of linear mapping in Banach spaces*, *Proc. Amer. Math. Soc.* 72(1978), 297-300.
- [44] Rassias, Th. M., The problem of S. M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.* 246(2000), 352-378.
- [45] Rassias, Th. M., On the stability of the functional equations and a problem of Ulam, *Acta Applicandae Mathematicae*. 62(2000), 23-130.

- [46] Rassias, Th. M. and Tabor J., Stability of Mappings of Hyers-Ulam Type, *Hardronic Press, Inc., Palm Harbor, Florida* 1994.
- [47] H. Shin'ya, Spherical matrix functions and Banach representability for locally compact motion groups. *Japan. J. Math. (N.S.)* 28(2002), No. 2, 163-201.
- [48] A. Redouani, E. Elqorachi, M. Th. Rassias The superstability of d'Alemberts functional equation on step 2 nilpotent groups, *Aequationes math.*, 74(2007), No. 3, 226-241.
- [49] H. Stetkær , Functional equations and matrix-valued spherical functions. *Aequationes Math.* 69(2005), No. 3, 271-292.
- [50] H. Stetkær Functional Equations and Spherical Functions. *Preprint Series 1994 No. 18, Matematisk Institut, Aarhus University, Denmark pp. 1-18.*
- [51] H. Stetkær, Functional equations on abelian groups with involution. *Aequationes Math.* 54(1997), No. 1-2, 144-172.
- [52] H. Stetkær, Functional equations on groups, Word Scientific, New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai, 2013.
- [53] H. Stetkær, d'Alemberts equation and spherical functions. *Aequationes Math.* 48 (1994), No. 2-3, 220-227.
- [54] H. Stetkær, Wilsons functional equations on groups. *Aequationes Math.* 49(1995), No. 3, 252-275.
- [55] Székelyhidi L., On a theorem of Baker, Lawrence and Zorzitto, *Proc. Amer. Math. Soc.* 84 (1982), 95-96.
- [56] Székelyhidi L., The stability of d'Alembert-type functional equations, *Acta Sci. Math. Szeged* 44 (1982), 313-320.
- [57] Székelyhidi L., On a stability theorem, *C. R. Math. Acad. Sc. Canada* 3 (1981), 253-255.
- [58] Ulam S. M., *A Collection of Mathematical Problems*, Interscience Publ. New York, 1961. *Problems in Modern Mathematics*, Wiley, New York 1964.

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